Amendments to the Drawings:

The attached sheets of drawings include changes to Figures 2 and 7A-7C. These sheets replace the original sheets that included Figures 2 and 7A-7C. In Figure 2, reference numbers 14, 16, 18 and 20 have been deleted, and lead lines have been added for Arm 1 and Arm 2. In Figures 7A-7C, reference numbers 14 and 16 have been replaced by reference numbers 14a and 16a, and reference numbers 14b ad 16b have been added.

Attachment: Replacement Sheet
Annotated Sheet Showing Changes

REMARKS

The claims pending in the subject application, claims 1-26, stand rejected under 35 U.S.C. 112. According to the examiner, there is a conflict in what is shown in Figures 1 and 2.

As discussed below, it is respectfully submitted that Figures 1 and 2 are consistent. However, to clarify what is shown in these figures, Figure 2 and Figures 7A-7C have been amended.

Figure 1 of the subject application shows a cantilever beam 14 mounted parallel to the y-axis and fixed to a support 18. The beam 14 bends with negligible extension, and the motion of the tip of the beam is substantially in the x direction. Thus, the beam 14 is called the x-arm (Arm 1). Also in Figure 1, a cantilever beam 16 is mounted parallel to the x-axis and fixed to a support 20. The beam 16 bends with negligible extension, and the motion of the tip of this beam is substantially in the y direction. Thus, the beam 16 is called the y-arm (Arm 2).

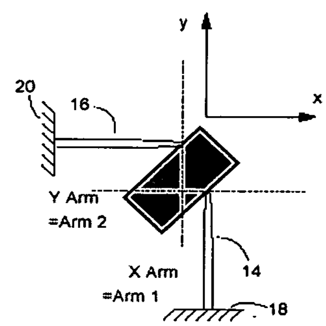
To simplify the analysis of cantilever beam stiffness, it is common to schematically represent a cantilever by a linear spring at the end of the beam, oriented perpendicular to the cantilever beam, and parallel to the beam bending direction. It is understood that this is an approximation which neglects the extension of the beam. This representation is common in undergraduate mechanical engineering curricula such as at Figure 1 of Glauser et al., *Mechanical and Aeronautical Engineering Senior Laboratory*, pp. 1-10. See, the section entitled "Modeling a Cantilever Beam." Reference can also be made to page 8 of *ASEN 2003: Introduction to Dynamics and Systems Spring 2002,* pp. 1-19. See, the section entitled "B.3 Modeling of Vibrations of Cantilever Beams by Elementary Beam Theory." Copies of both of these publications are attached.

Figure 2 schematically represents the equivalent stiffness of the beam 14, the x-arm, and the beam 16, the y-arm. The bending stiffness of the beam 14 is substantially along the x direction and is represented by a virtual spring element 14a. The high stiffness which prevents

extension of the beam 14 in the y direction is represented by a virtual constraint element 14b. The bending stiffness of the beam 16 is substantially along the y direction is represented by a virtual spring element 16a. The high stiffness which prevents extension of the beam 16 in the x direction is represented by a virtual constraint element 16b.

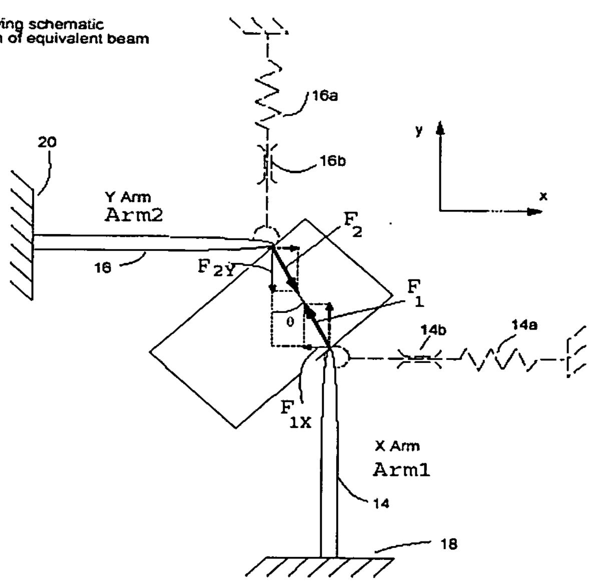
By way of further explanation, a simplified plan view of the system of Figure 1 of the subject specification is shown below:

Top View of Figure 1



Combining this view of Figure 1 with that of Figure 2, as illustrated below, shows the schematic representation of the equivalent beam stiffness:

Figure 2 showing schematic representation of equivalent beam stiffness.



Thus, as can be seen from the subject specification, as well as from the above explanation, what is depicted in Figures 1 and 2 is consistent with the operation of the claimed apparatus for manipulating an object.

In view of the foregoing, it is submitted that all the claims are now in condition for allowance. Accordingly, allowance of the claims at the earliest possible date is requested.

If prosecution of this application can be assisted by telephone, the Examiner is requested to call Applicants' undersigned attorney at (510) 495-3206.

Please apply any other charges or credits to Deposit Account No. 500388.

Respectfully submitted,
BEYER WEAVER & THOMAS, LLP

Date: 1/28/04

William J. Egan, III
Reg. No. 28,411

P.O. Box 778
Berkeley, CA 94704-0778

Annotated Sheet Showing Changes

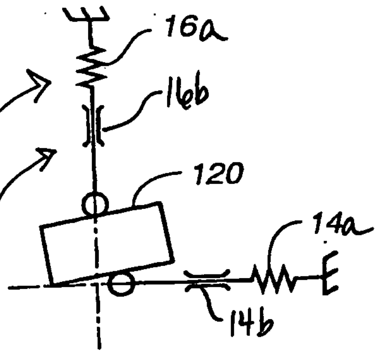


FIG._7A

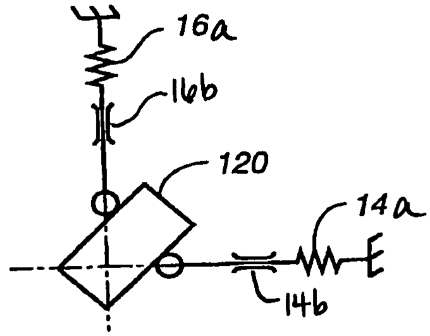


FIG._7B

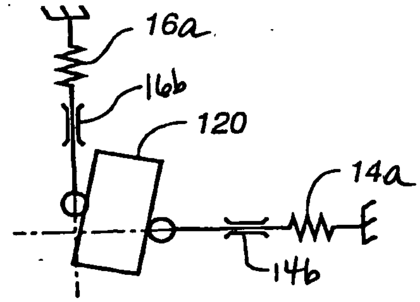


FIG._7C

Reference numbers 14 and 16 replaced with reference numbers 14a and 16a, and reference numbers 14b and 16b have been added.

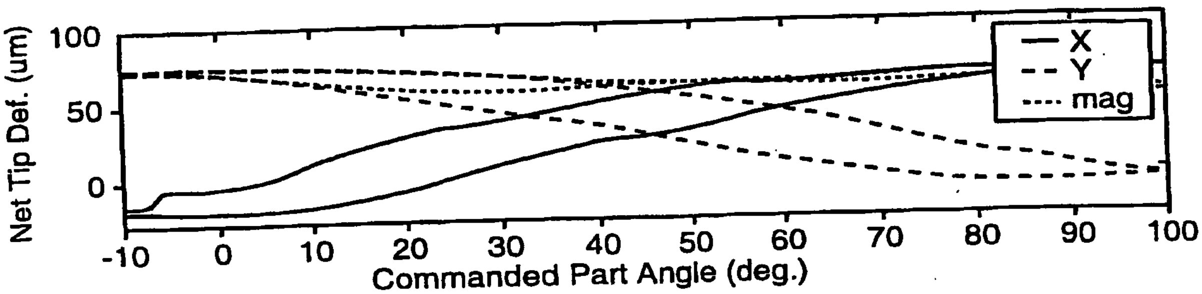


FIG._8A

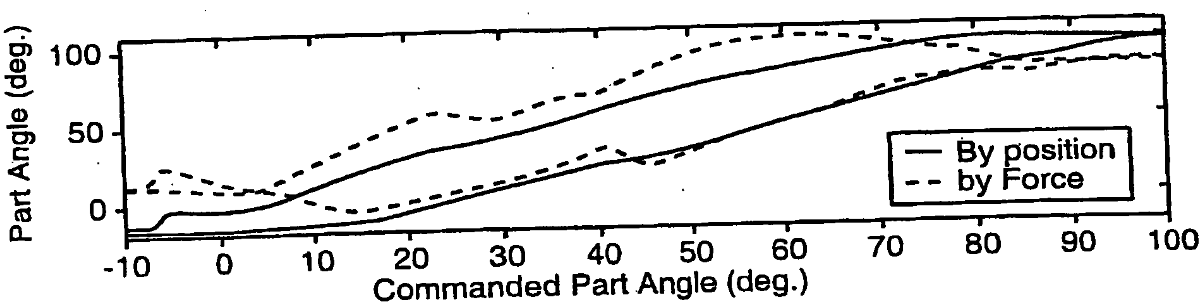


FIG._8B

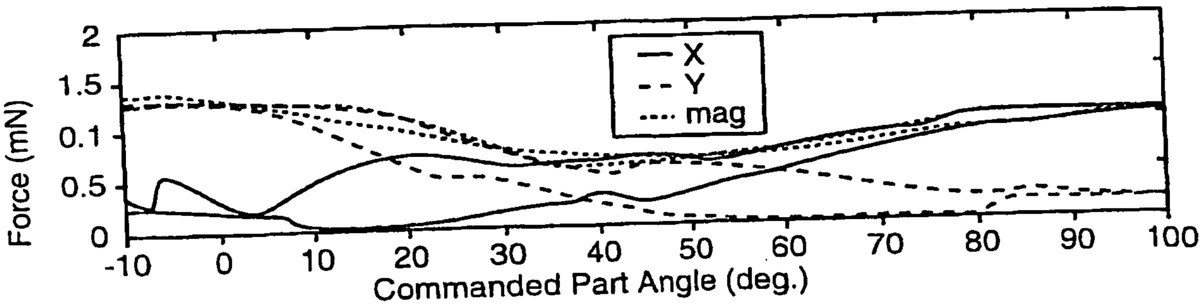


FIG._8C

Annotated Sheet Showing Changes

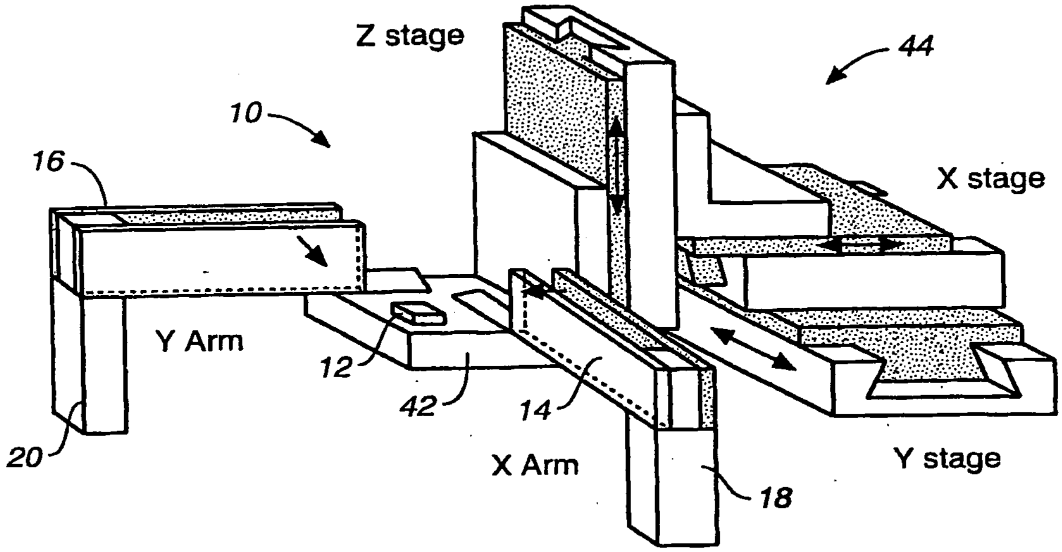


FIG._1

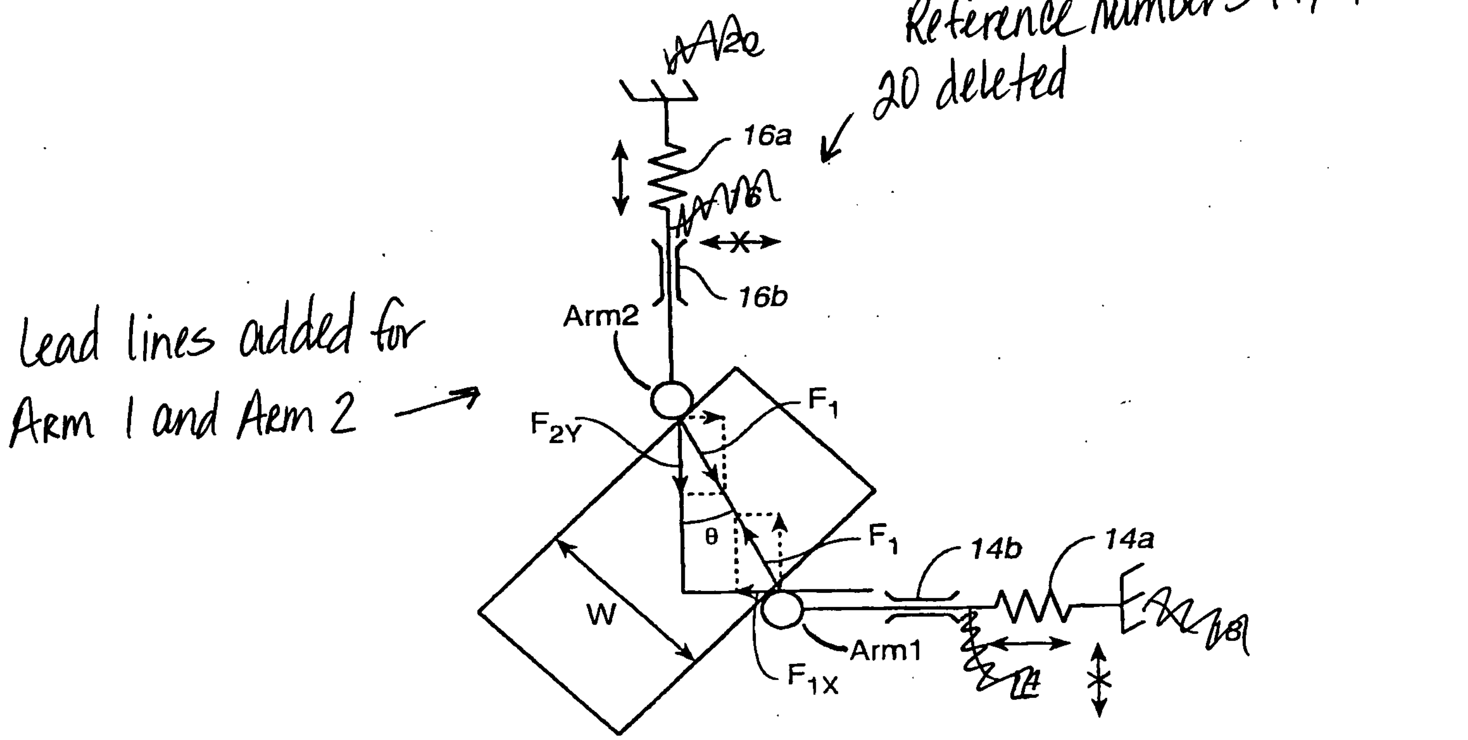


FIG._2

Mechanical and Aeronautical Engineering Senior Laboratory

Dr. Mark Glauser - Course Instructor
266 CAMP Building
(315) 268-6683 phone

Charles Tinney - Teaching Assistant
265/269 CAMP Building
(315) 268-3825 phone

Matthew Duquette - Teaching Assistant
252 CAMP Building
(315) 268-7683 phone

Introduction to Experiment 1: Vibrations

From jackhammers to buildings, automotive suspension systems to aircraft gas turbine engines, it is obvious to even the most casual observer that vibrations are extremely important. In this experiment you will examine the vibrations of the spring mass, damper system. In part 1 you will look at the effect of sinusoidal excitation on a cantilever system with small damping. The trust will be to gain an understanding of the natural frequency of a system. You will vary the mass of the beam, hence effecting its frequency response characteristics. In part 2 of this lab you will study the effect of damping on the frequency response characteristics. This part of the lab provides insight into how the engineer and scientist can use damping to minimize the negative effects of vibrations. Before you begin this lab you should review second order differential equations with constant coefficients. Keep in mind what quantities in the governing second order differential equation for the spring mass damper system that you are changing throughout the experiment. This should be discussed in some detail in the lab write-up.

Part 1: Small Damping

Required Equipment:

- Shaker Table

- Accelerometer
- Steel bar
- Charge Amplifier (I.C.P)
- Instrumented Hammer
- Dell Pentium III 800 with National Instruments LabVIEW 5.1
- Measuring tape or ruler
- Oscilloscope

Experiment Apparatus:

The apparatus consist of a steel cantilever beam mounted on a shaker table, which is capable of producing frequencies of up to 60 Hz. An accelerometer is mounted on the free end of the beam, and is used in conjunction with a charge amplifier and oscilloscope to monitor the beam's response. In addition, extra masses are available that can be added to the cantilever to alter its response.

In this experiment, the oscilloscope is used to monitor the electrical signal, which is produced by the accelerometer/charge amplifier.

The oscilloscope trace is actually a graph of voltage vs. time, which can be translated or scaled using the oscilloscope adjustments. The trace can be "stretched" or "shrunk" in either the horizontal or vertical directions by adjusting the voltage (vertical) or time (horizontal) scales. On the oscilloscope, these scales are expressed as either voltage or time per division. For example, if the time scale is set to 10 ms per division, the horizontal dimension of each square block on the display grid corresponds to 10 milliseconds.

The LabVIEW software will be used to acquire the experimental data. The format of this file is found following the "Topics for Discussion".

Elementary Theory:

A Simple Spring-Mass System

Many oscillating systems can be modeled as a spring-mass system using as differential equation of motion. The displacement, y(t), of such systems can be found using

$$my'' + cy' + ky = F(t),$$

where m is the mass of the object, c is the damping coefficient, k is the spring constant, and F(t) is some forcing function. Each term in this expression is actually a time-dependent force: my,, is the inertial force, cy, is the frictional force, and ky is the spring force. Modeling the damping of a system in this way assumes that the damping force is proportional to the velocity of the mass; this is called viscous damping. For a freely vibrating system (F(t)=0), the damping coefficient of the system can be found experimentally using

$$c = \frac{1}{2\pi} \ln \frac{Y_1}{Y_2},$$

where Y_1 and Y_2 are the values of any two consecutive maximum displacements that are one cycle apart.

The natural frequency of a system is the frequency at which an undamped system will freely vibrate, and can be calculated by using

$$\omega_n \left(\frac{rad}{s}\right) = \sqrt{\frac{k}{m}}$$

$$f_n(Hertz) = \frac{\omega_n}{2\pi}$$

<u>Modeling a Cantilever Beam</u>

The cantilever beam is an example of a system, which can be modeled as a simple spring-mass system. In order to model the vibration of the cantilever beam, the end of the beam is chosen as a reference point at which the characteristics and response of the beam are measured (see Figure 1). An equivalent system is then built that has a response, y(t), that is identical to that of the actual system.

The spring constant, k, of the equivalent system is identical to that of the cantilever beam, and can be calculated quite easily using beam deflection formulae. Calculation of the equivalent mass, however, is much more difficult because all points along the beam,s length do not have the same response as the end of the beam. This means that the

equivalent mass, m, can not be determined simply by adding the masses m_{beam} and m_{end}, but must be found by equating the energies of the two systems as they vibrate. This type of analysis is called lumping. A discussion of this technique can be found in J. E. Shigley, *Mechanical Engineering Design*, McGraw Hill, 1977.

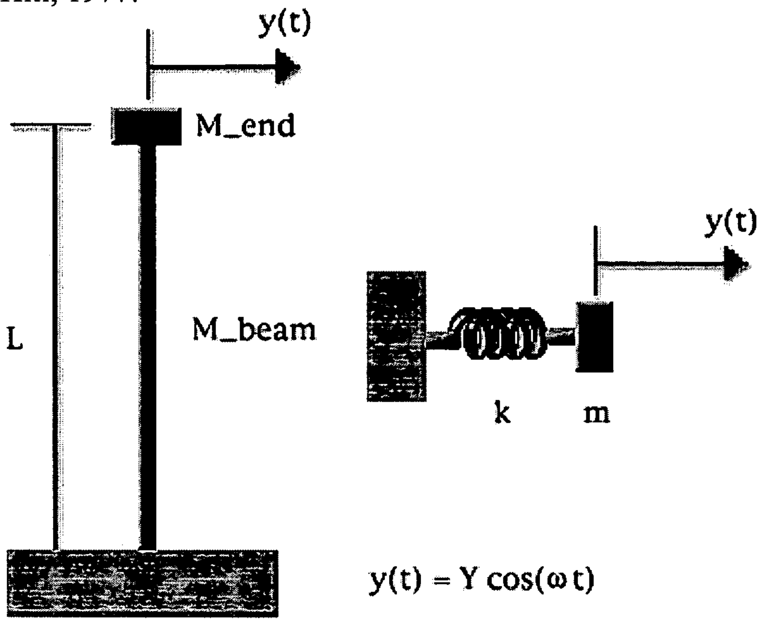


Figure 1: Modeling the cantilever as a simple spring / mass system

The energy, U, stored in the simple spring-mass system is

$$U = \frac{\omega^2}{2} Y^2 m \quad ,$$

where ω, is the radial frequency and Y is the Mass,s amplitude. To find the energy stored in the cantilever, the energies of both the beam and the mass at the end of the beam must be considered.

$$U_{system} = U_{beam} + U_{end}$$

$$U_{system} = \frac{\omega^2}{2} \int_{x=0}^{x=l} \{Y(x)\}^2 dm_{beam} + \frac{\omega^2}{2} Y^2 m_{end}$$

Performing the integration, the energy stored in the actual cantilever system is found to be

$$U_{system} = \frac{\omega^2}{2} Y^2 \{\frac{33}{140} m_{beam} + m_{end}\}$$

Hence, by comparing equations (1) and (2), we see that the real cantilever system can be modeled as a simple spring-mass system with a mass that is given by

$$m = \frac{33}{140} m_{beam} + m_{end}$$

Procedure:

Measure the dimensions of the beam for use in calculation of the theoretical natural frequency. Also, record the mass of the extra weights that are to be attached to the end of the beam during the experiments. The mass of the accelerometer is 25g. Be sure to measure only the portion of the beam which will be vibrationg. Do not include the portion of the beam clamped between the jaws of the vice.

The Shaker Table Test

In this experiment, the bars forced frequency is investigated. When the forcing frequency matches the beam's natural frequency, resonance is observed.

- Make sure that the bar is clamped securely in the vice
- Turn on the shaker table (Note: the switch for the motor is on a small switch box facing the wall)
- Increase the table frequency from 0 to 60 Hz without stopping and then proceed back to 0 Hz.
- Record the resonance frequency. Allow the table to operated at this frequency for only a very short time!
- Turn off the shaker table
- Repeat the above procedure with each of the extra masses attached to the free end of the beam. Make sure that each mass is secured firmly.

The Accelerometer Test

In this test, the natural frequency and damping of the bar,s free response is investigated.

- Ensure that the I.C.P. charge amplifier is connected to the accelerometer and to the input for channel 1 on the oscilloscope via a "T" connector. A separate BNC cable connects the interface box to the "T" connector.
- Turn on the I.C.P., the oscilloscope and the computer. (Note: the amolifier should be turned on prior to your arrival by the TA, but if not, allow 5-10 minutes for the electronics to warm up)
- Make sure that the needle on the I.C.P. unit is near the center of the green band.
- For monitoring purposes, set the oscilloscope for the following:
    - Non-storage
    - Internal source
    - Auto trigger off
    - Channel 1
    - A.C.
    - Slope (+)
- Adjust the vertical position so that the trace is on the centerline.
- Adjust the time and voltage scales so that a clean response results when the bar is hit with the rubber hammer. Make sure that the end of the time scale knob is turned to its "X1" setting.
- Run LabVIEW VI and follow the instructions on the screen.

Exercises

- Calculate the theoretical natural frequency of the beam with and without the extra masses attached.
- Calculate the damping coefficients for each of the cantilever systems.
- Calculate the natural frequency using the data from the accelerometer test.
- On the same graph, plot the theoretical natural frequency vs. mass curve for the cantilever beam system and all of the experimental data points.

Topics for Discussion

- Describe the system,s response (i.e. the amplitude and frequency) to different forcing frequencies.
- Compare the various measures of natural

frequency.
- How does the natural frequency vary with mass?
- Discuss the type of damping that is present in the cantilever.

The LabVIEW Vi will generate a tab delimited ASCii text document in the following format:

Vibration Experiment - Cantilevered Beam
Friday, February 05, 1999
3:03 PM

Beam Dimensions:
Length of Beam (mm): nnnE+n
Width of Beam (mm): nnnE+n
Thickness of Beam (mm): nnnE+n
End Weight (kg): nnnE+n

System Response (by iteration):

Frequency(Hz):	Phase(f):	Voltage(f):
nnnE+n	nnnE+n	nnnE+n
nnnE+n	nnnE+n	nnnE+n
nnnE+n	nnnE+n	nnnE+n
:	:	:
nnnE+n	nnnE+n	nnnE+n

Time History (from impact):

Time (sec):	Voltage (t):
nnnE+n	nnnE+n
nnnE+n	nnnE+n
nnnE+n	nnnE+n
:	:
nnnE+n	nnnE+n

System Response (from impact):

Frequency (Hz):	Phase(f):	Voltage(f):
nnnE+n	nnnE+n	nnnE+n
nnnE+n	nnnE+n	nnnE+n
nnnE+n	nnnE+n	nnnE+n
:	:	:
nnnE+n	nnnE+n	nnnE+n

Part 2: Effects of Damping

Required Equipment:

- Model 9210 Drive Harmonic Motion Analyzer
- 3 cylinders
- mass disk
- water
- glycerin

Procedure:

Before all test, make sure that the analyzer is leveled and zeroed. Leveling is important to keep the rod from striking the table or rubbing against the housing. To level, adjust the feet on the bottom of the analyzer.

After leveling, reset the zero phase reference by [1] rotating the pulley on the back of the unit until the scale is horizontal and [2] adjusting the height off the mass (using the fine adjustment knob at the top of the pulley and the gross adjustment by moving the tape) to the point where the red light (which senses the mass amplitude scale) is barely glowing. Reset the zero phase reference before each new test.

Allow the system to come to a steady state before recording any data. The results from each test should correspond to two steady state equations:

Forcing function = $A_f \sin(2\pi F_f t)$

Mass displacement = $A_m \sin(2\pi F_f t + \theta)$

where A_f is the forcing amplitude, A_m is the mass amplitude, F_f is the forcing frequency, and θ is the phase angle. For all test record the following parameters.

- A_f - forcing amplitude, which is determined by reading the scale on the back of the machine.
- F_f - forcing frequency
- A_m - mass amplitude
- P_m - mass period
- θ - phase angle (lag: $\theta < 0$; lead $\theta > 0$)

F_f, A_m, and P_m are read off the digital readout by changing the switch.

Test 1: Time Response

Set up the analyzer with an empty cylinder and no extra mass. Choose a forcing amplitude and frequency of 1 Hz or below so that the rod does no hit the bottom of the cylinder and the canes do not

come out of the cylinder. After the system comes to a steady state, record all of the parameters listed above. Repeat this procedure first with water in the cylinder, then with glycerin making sure to keep the some forcing amplitude and frequency. Note that the drive can be turned off without adjusting the frequency. Record all parameters for each case.

Add on mass disk to the damping rod (not on the vanes) and reset the zero phase reference. With the same forcing amplitude and frequency, repeat for air, water, and glycerin, recording all parameters. (Note: only use one mass)

Answer the following questions in reference to the recorded parameters.

1. Describe the effects (if any) on the response of the system due to the different substances in the damping cylinder.
2. Describe the effects (if any) on the response of the system due to the increase in mass.
3. On two graphs of displacement vs. time, plot the forcing function (which is the same for both graphs) and all 3 responses for a particular mass, using the given equations and the values obtained experimentally (not data points).

Test 2: Frequency Response

Remove the extra mass and reset the zero phase reference. By adjusting the frequency, find and record the approximate value for the natural frequency of the system in air. Please do not le the rod bang on the table more than necessary.

With the same forcing amplitude as before, record values for all parameters for six different forcing frequencies: three below the natural frequency and three above it. Repeat for both water and glycerin. (Note: make sure you're not too close to the natural frequency)

Approximate values for natural frequency :

Air:

Water:

Glycerin

Plot (A_m/A_f) vs. ω and θ vs. ω for each of the three damping conditions (two graphs with three plots each).

From these plots, refine your estimate of the natural frequency. This is where A_m tends toward infinity for no damping and where $\theta = -90°$.

Revised natural frequencies:

Air:

Water:

Glycerin

Replot your results on two graphs, (A_m/A_f) vs. $\omega*$ and θ vs. $\omega*$, where $\omega* = \omega/\omega_n$. Both of these graphs will have six plots ^ make them as neat and as large as possible.

Describe in terms of the recorded parameters what effect the forcing frequency has on the response of the system. Be sure to include the relevant uncertainty analysis for the measured quantities (see notes in lab manual).

This site is maintained by Jeffrey Taylor

Revised: August 3, 1999.

Appendix A. Natural Vibration Modes and Mode Shapes

A structural element can vibrate with different frequencies depending upon the nature of excitations. For example, when a cantilever beam is placed on the shaker and subjected to varying frequency excitations, one observes that at particular excitation frequencies the beam vibrates violently. These violent frequencies are called resonant modes or natural vibration modes. One purpose of vibration analysis and vibration experiment is to identify the resonant vibration frequencies so that the expected operational conditions would not trigger those modes.

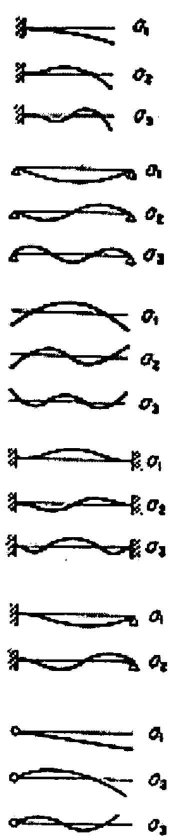
V. Uniform Beams

(Transverse or bending vibrations)

The same general formula holds for all the following cases,

$$\omega_n = a_n \sqrt{\frac{EI}{\mu_1 l^4}}$$

where EI is the bending stiffness of the section, l is the length of the beam, μ_1 is the mass per unit length $= W/gl$, and a_n is a numerical constant, different for each case and listed below



Cantilever or "clamped-free" beam	$a_1 = 3.52$ $a_2 = 22.0$ $a_3 = 61.7$ $a_4 = 121.0$ $a_5 = 200.0$
Simply supported or "hinged-hinged" beam	$a_1 = \pi^2 = 9.87$ $a_2 = 4\pi^2 = 39.5$ $a_3 = 9\pi^2 = 88.9$ $a_4 = 16\pi^2 = 158.$ $a_5 = 25\pi^2 = 247.$
"Free-free" beam or floating ship	$a_1 = 22.0$ $a_2 = 61.7$ $a_3 = 121.0$ $a_4 = 200.0$ $a_5 = 298.2$
"Clamped-clamped" beam has same frequencies as "free-free"	$a_1 = 22.0$ $a_2 = 61.7$ $a_3 = 121.0$ $a_4 = 200.0$ $a_5 = 298.2$
"Clamped-hinged" beam may be considered as half a "clamped-clamped" beam for even a-numbers	$a_1 = 15.4$ $a_2 = 50.0$ $a_3 = 104.$ $a_4 = 178.$ $a_5 = 272.$
"Hinged-free" beam or wing of autogyro may be considered as half a "free-free" beam for even a-numbers	$a_1 = 0$ $a_2 = 15.4$ $a_3 = 50.0$ $a_4 = 104.$ $a_5 = 178.$

Table 1 Modes and Mode Shapes of Beam Vibrations
(Source: Mechanical Vibrations by J. P. Den Hartog, Dover Pub., 1985, p.432)

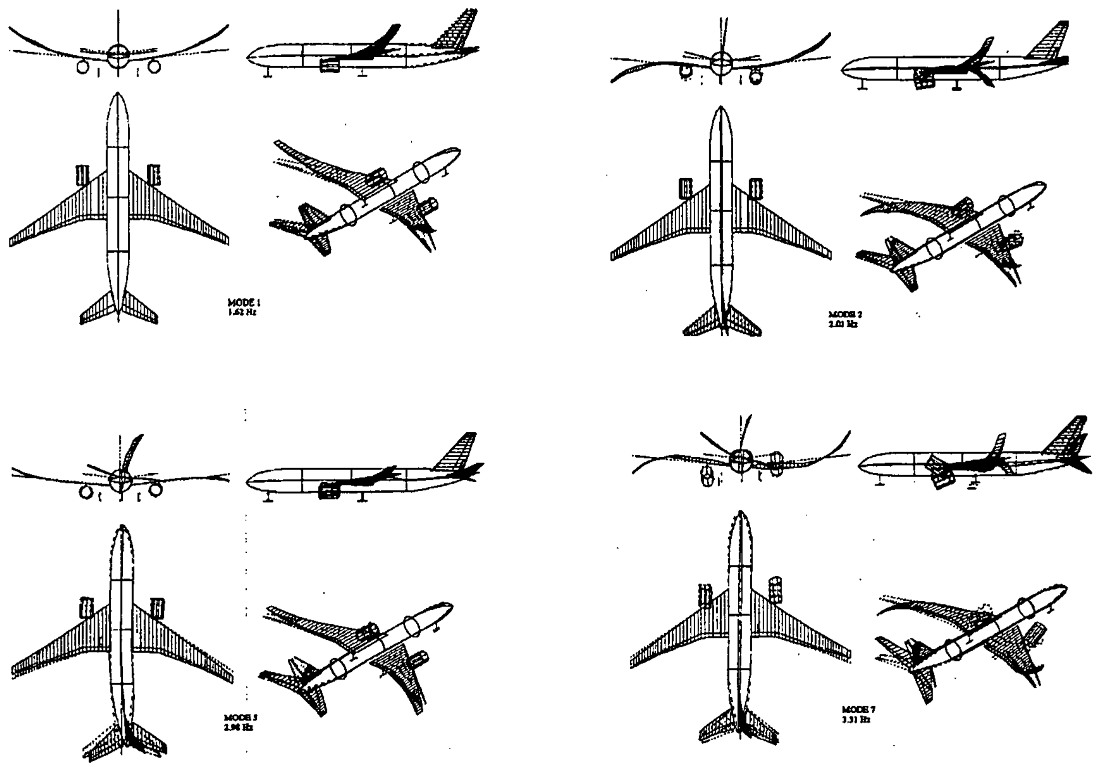


Fig. 1 Selected Vibration Modes and Mode Shapes of a Boeing 777 Plane
(Source: Dr. Michael Mohaghegh of Boeing Co.)

Table 1 illustrates the analytical frequency formulas for various boundary conditions anddifferent mode shapes. Note that the natural frequency of a uniform is expressed as

$$\omega_n = a_n \sqrt{\frac{EI}{\mu \ell^4}} \tag{1}$$

where a_n is a coefficient given in Table 1, E is Young's modulus of the beam material, I is the second area moment which becomes $I_{rectangle} = bh^3/12$ for a beam of rectangular cross section with its width b and thickness h, and μ is the mass per unit length, that is, $\mu = \rho A = \rho bh$ for a rectangular cross section.

In order to gain insight into the overall vibration characteristics, selected modes and mode shapes of a Boeing 777 Airplane are shown in Fig. 1. Once sufficient natural vibration modes of the model

airplane are identified, it is important to understand in what ways the vibration characteristics of substructural components contribute to the overall vibrations. To this end, we disassemble the plane and carry out component vibration testing of the wing, the fuselage, and the tail boom separately. In this experiment, a cantilever beam is used to aid you to gain insight into how component vibration modes and mode shapes would influence the overall vibrations. In addition, Figure 2 illustrates first five natural frequencies and mode shapes of a VC-10 plane, ranging from 1.85 to 4.28 Hz. Note that assembled wings, fuselage and tail boom may or may not vibrate with the elemental natural modes. In addition, the *in-flight* vibration modes would not be the same as the *on-ground* airplane modes.

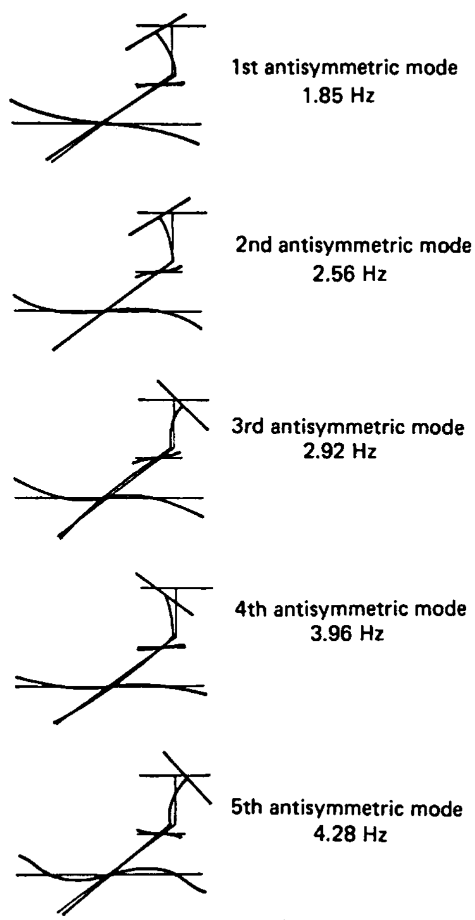


Fig. 2 Vibration Modes and Mode Shapes of a VC-10 Plane *in flight condition*
(Source: Vibration by R. E. D. Bishop, Cambridge University Press, 1979)

APPENDIX B. INTRODUCTION TO VIBRATION THEORY

B.1 Introduction

VIBRATIONS is a terminology used to describe a broad range of phenomena, both natural and man-made, ranging from the unceasing flapping wings of a honeysuckle, the swaying back and forth of trees during storm wind, to beautiful violin sounds and the rattling of the steering wheel of a car in motion. Our bodies are an embodiment of vibratory phenomena which Prof. R. E. B. Bishop[1] eloquently summarized in his 1962 Christmas Lectures at the Royal Institution, London: "After all, our hearts beat, our lungs oscillate, we shiver when we are cold, we sometimes snore, we can hear and speak because our eardrums and our larynges vibrate. . . . We move by oscillating our legs. We can not even say 'vibration' properly without the tip of the tongue oscillating. And the matter does not end there - far from it. Even the atoms of which we are constituted vibrate."

In this course, we will primarily focus on the vibrations of mechanical systems, or *engineering vibrations*, that are man-made devices, machines, and transportation systems such as automobiles, airplanes and satellites as every successful engineering design must address vibration problems. The first design category is to create desirable vibrating conditions. This includes clocks, the speed of constant turbine rotations, a concrete mixer, the spin rate of spin-stabilized satellites, and electric massagers, among others. The second design category is to alleviate vibrations altogether. This include the bumpy riding quality of a car or a airplane, wind-induced vibrations of bridges, excessive noise from washer and dryer, and fatigue failure of critical machine components due to prolonged vibrations. It is this vibration alleviation or vibration control that constitute a major task to practicing engineers.

The vibrations of mechanical systems are caused by sudden or continuous disturbances, such as aerodynamic forces on the airplanes, oscillations of internal combustion engines in automobiles, motors in a washer. Qualitatively speaking, the energy contained in the disturbances are transmitted to the mechanical systems and finds its way to propagate throughout the system in a way analogous to the way flood water finds its way through the flooded region. Unlike the flooding water that flows usually into lower altitude, however, the energy carried by mechanical vibrations from one part of the system may reverse its transmission path and reverse its flow path. This can be seen when one stretches the spring hanging vertically from the ceiling and let it go for *its free vibrations* to continue. In other words, oscillations of a mechanical system means that the system energy can move back and forth.

Therefore, the designer must assess whether any component of the mechanical system break during the initial violent disturbance stages, subsequent steady-state vibrations, degradations of system performance during the post-disturbance period, and potential fatigue failure due to prolonged vibrations. These four distinct vibratory response characteristics are referred to *transient vibrations*, *harmonic oscillations*, *free vibrations*, and *fatigue cycles*. Sudden gust winds acting on an airplane in flight is of transient nature, and the subsequent post-gust vibrations are of free vibrations. Fatigue cycles are primarily due to normal operational loads. For example, a woman's heart would have vibrated 2.95 billion times assuming 72 beats per minutes for an average longevity of 78 years. A mechanical component in an airplane whose fundamental frequency of 72 Hertz would have vibrated

for about 370 million times when the plane has flown about 87000 hours (8 hours everyday for 30 years). A tragic feature of fatigue failures in both humans and machine components is that they lead to catastrophes when fatigue failures occur.

B.2 Frequency of a Single Mass and Spring System

Consider a single mass m attached to a spring k that is subjected to an applied force $f(t)$ as shown in Fig. 1. Applying Newton's second law, one obtains the equation of motion for this system as

$$m\,\ddot{x}(t) + k\,x(t) \;=\; f(t) \tag{2}$$

where $x(t)$ is measured from the static equilibrium condition under its own weight. Of particular interest is the nature of its free vibration, that is, the vibration of the spring-mass system without applied force, with an initial displacement $x(0)$ and/or an initial velocity condition $\dot{x}(0)$. The differential equation to be solved can be written as

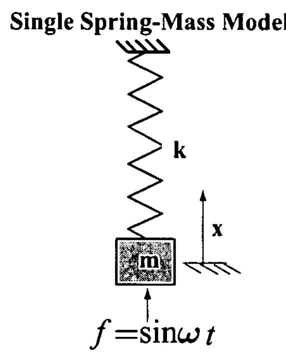


Fig. 1 Single DOF Spring-Mass Model

$$m\,\ddot{x}(t) + k\,x(t) \;= 0, \quad \text{with} \quad x(0) = x_0,\; \dot{x}(0) = \dot{x}_0 \tag{3}$$

We assume the solution of this equation in the form

$$x(t) = A\;\cos pt + B\;\sin pt \tag{4}$$

where A p and B are constants to be determined. Substituting

$$\ddot{x}(t) = -p^2(A\;\cos pt + B\;\sin pt\,) \tag{5}$$

and (4) into (3), we obtain

$$(-p^2\,m + k)\;(A\;\cos pt + B\;\sin pt\,) = 0 \tag{6}$$

As A and B are not necessarily zero, one must have the term in the parenthesis to be zero:

$$(-p^2\,m + k) = 0 \quad \Rightarrow \quad p = \sqrt{k/m} \qquad (rad/sec) \tag{7}$$

Substituting this into (4), we obtain

$$x(t) = A\ \cos\sqrt{k/m}\,t + B\ \sin\sqrt{k/m}\,t \tag{8}$$

The integration constants A and B can be determined by using the initial conditions ($x(0) = x_0, \dot{x}(0) = \dot{x}_0$) as follows:

$$\begin{aligned} x(0) = x_0 = A\ \cos 0 + B\ \sin 0 \quad &\Rightarrow \quad A = x_0 \\ \dot{x}(0) = \dot{x}_0 = \sqrt{k/m}\,(-A\ \sin 0 + B\ \cos 0\,) \quad &\Rightarrow \quad B = \dot{x}_0/\sqrt{k/m} \end{aligned} \tag{9}$$

Finally, substituting this into (3) we obtain

$$\boxed{x(t) = x_0\ \cos pt + \dot{x}_0\ \sin pt, \quad p = \sqrt{\frac{k}{m}}} \tag{10}$$

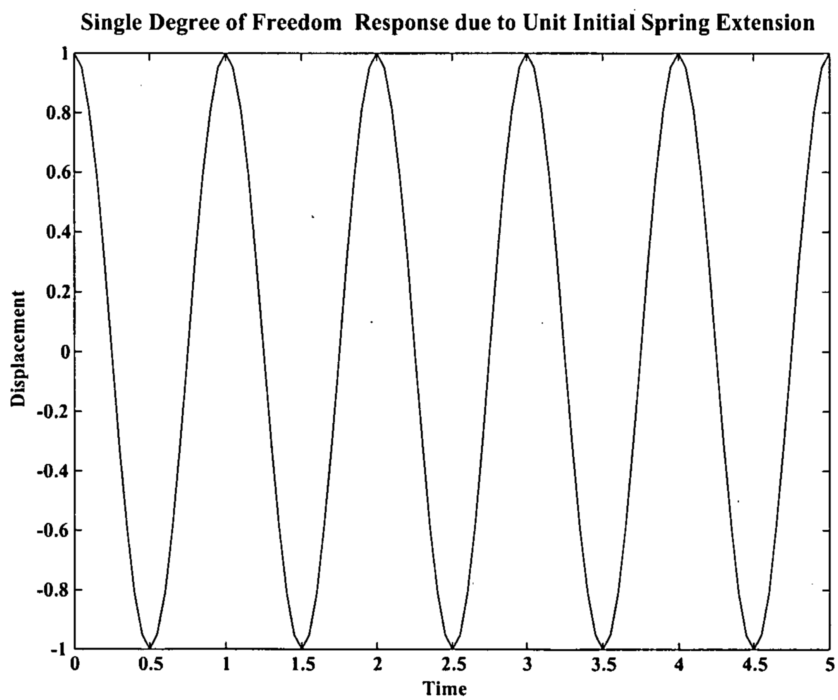


Fig. 2 Response of Single DOF Spring-Mass System to Unit Initial Displacement Condition

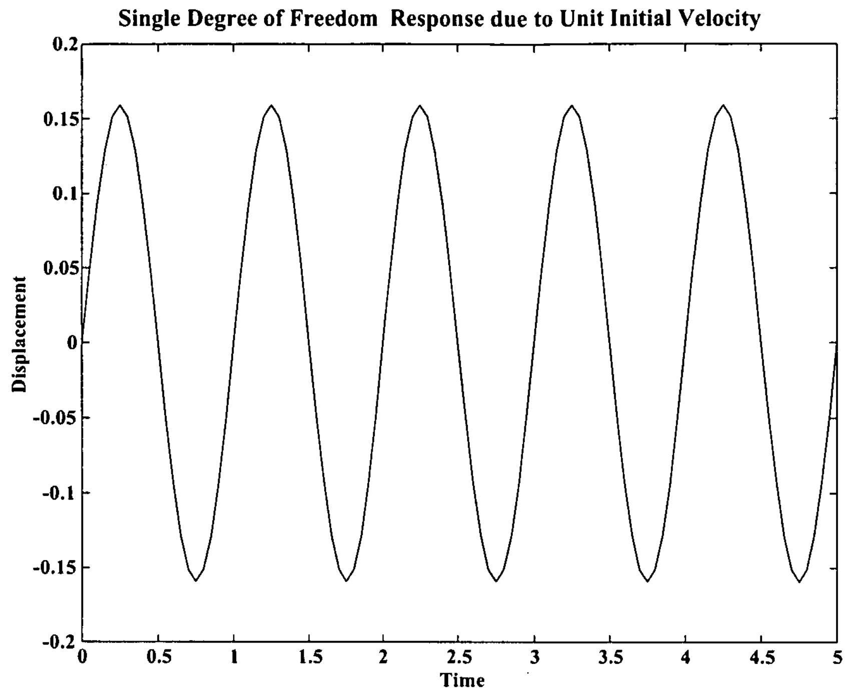


Fig. 3 Response of Single DOF Spring-Mass System to Unit Initial Velocity Condition

Figure 2 shows the response $x(t)$ when the spring is extended unit displacement and let to-and-fro motions (vibrations) continue. In Fig. 3 we illustrate the response $x(t)$ vs. time when a unit velocity is applied. Note that the unit displacement response and the unit velocity response are 90^0 out of phase. The frequency of the model system is given by

$$p = \sqrt{\frac{k}{m}} \qquad (rad/sec) \tag{11}$$

The frequency p in (rad/sec) unit can be converted into to Hz $(Hertz)$ by:

$$p = 2\pi f \qquad \Rightarrow \quad f = \frac{p}{2\pi} \qquad (Hz) \tag{12}$$

Note that, when the mass is fixed, the frequency is proportional to

$$f \propto \sqrt{k} \tag{13}$$

In practice, whenever the spring constant is altered, by using thicker beams or heavier springs, the mass is also changed.

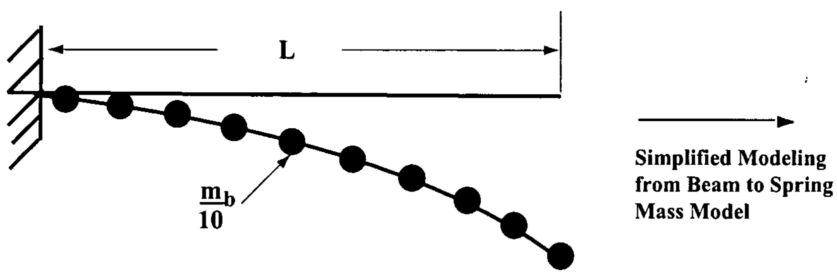
An important fact is that as long as the ratio of the spring to the mass k/m is constant, the system frequency remains constant. This provides not only a design criterion for vibrations, but also provides the possibility of scale-model testing by constructing a laboratory model according to the relation:

$$p^2 \propto \frac{K}{M} \propto \frac{k}{m} \tag{14}$$

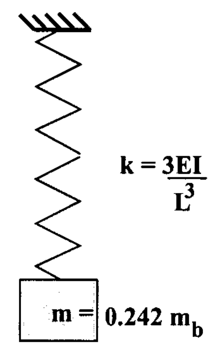
where (K, M) and (k, m) refer to the real structure and the laboratory scale-model, respectively, for which one usually selects $M >> m$.

B.3 Modeling of Vibrations of Cantilever Beams by Elementary Beam Theory

Consider a cantilever beam that is subjected to a tip load P. If the tip load is suddenly removed, the beam will continue to vibrate, exhibiting its free vibration mode as shown in Fig. 4a.



a) First Vibration Mode Shape of a Cantilever Beam: A cantilever beam can be thought of consisting several discrete masses as shown above. Note that the tip mass velocity is the greatest while that of the mass closest to the fixed end is the slowest. Thus, the contribution of each mass to the system kinetic energy varies according to the vibration mode shape.



b) Equivalent Spring-Mass Model of First Mode Cantilever Vibration

Fig. 4 Modeling Cantilever Beam by an Equivalent Spring-Mass System

The static deflection of the beam, $w(x)$, along the beam length is given by

$$y(x) = \frac{P}{6EI_{zz}}(x^3 - 3Lx^2), \quad 0 \le x \le L \tag{15}$$

where y, P, E, I_{zz}, and L are deflection along the beam span x, the tip load, Young's modulus, the area moment of inertia of the beam cross section, and the beam length, respectively. Note that the beam deflection along the beam span is of a cubic shape.

At the tip end, we obtain with $x = L$ the following force vs. deflection relation:

$$P = k\,\delta, \quad k = \frac{3EI_{zz}}{L^3}, \quad \delta = y(L) \tag{16}$$

where δ is the tip deflection and k is the corresponding spring constant.

If the beam vibrates according to this particular characteristic deflection shape or commonly designated as *mode shape*, we can express the dynamics of this vibration characteristic or *mode*

$$m\,\ddot{\delta}(t) + k\,\delta(t) = P, \quad k = \frac{3EI_{zz}}{L^3} \tag{17}$$

It turns out that the mass m in the above equation is not equal to the mass of the total cantilever beam $m_b = \rho AL$. This is because the spring-mass equation (17) is an idealization of a cantilever beam as shown in Fig. 4b. The determination of the modal mass can be obtained from the system kinetic energy equivalence considerations. Note that the kinetic energy of the spring-mass system is given by

$$T = \tfrac{1}{2} m\, \dot{\delta}^2(t) \tag{18}$$

In order to obtain the modal mass m, we first express the kinetic energy T in terms of the distributed velocity of the beam, $\dot{y}(t)$. We then express $\dot{y}(x,t)$ in terms of the tip velocity $\dot{\delta}(x,t)$. This is derived below in a step-by-step manner:

- First, express from (16)

$$\frac{P}{EI_{zz}} = \frac{3\delta}{L^3} \tag{19}$$

- Second, substitute this into (15) to obtain

$$y(x,t) = \frac{\delta}{2L^3}\,(x^3 - 3Lx^2) \tag{20}$$

- Third, differentiate the above equation to obtain the velocity expression

$$\dot{y}(x,t) = \frac{\dot{\delta}}{2L^3}\,(x^3 - 3Lx^2) \tag{21}$$

- Fourth, obtain the kinetic energy of the beam by

$$T = \tfrac{1}{2}\int_0^V \dot{y}^2 \rho dV = \tfrac{1}{2}\int_0^L \dot{y}^2 \rho A\, dx \tag{22}$$

- Fifth, substituting $\dot{y}$ from (21) into the above equation, we obtain

$$T = \tfrac{1}{2}\int_0^L \dot{y}^2 \rho A\, dx = \tfrac{1}{2}\int_0^L (\frac{\dot{\delta}}{2L^3})^2\,(x^3 - 3Lx^2)^2 \rho A\, dx \tag{23}$$

  which, when the indicated integration is carried out, yields

$$T = \tfrac{1}{2}(\frac{33\rho AL}{140})\,\dot{\delta}^2 = \tfrac{1}{2} m\dot{\delta}^2 \tag{24}$$

The last expression in the preceding equation is realized by making the total kinetic energy of the beam equal to the kinetic energy of the spring-mass system.

Therefore, the modal mass m is obtained to be

$$m = (\frac{33\rho AL}{140}) = (\frac{33\, m_b}{140}) = 0.2357 m_b, \quad m_b = \rho AL \tag{25}$$

- Finally, the modal equation (16) becomes

$$m\,\ddot{\delta}(t) + k\,\delta(t) = P, \quad m = (\frac{33\rho AL}{140}), \quad k = \frac{3EI_{zz}}{L^3} \tag{26}$$

The vibration frequency of this modal equation is thus given by

$$\omega_1 = \sqrt{\frac{k}{m}} = a_1\sqrt{\frac{EI_{zz}}{\mu L^4}}, \quad a_1 = 3.52^{\#}, \quad \mu = \rho A \tag{27}$$

Equation (24) gives $a_1 = 3.5675$ which has about 1.35% error compared with the value obtained by a more advanced theory (Mechanical Vibrations by J. P. Den Hartog, pp.429-433).

Finally, often in a simplified modeling of complex structures by a beam, a concentrated mass is introduced along the beam span. For example, fuel tanks along the wing span may be modeled as concentrated masses. Figure 5 represents an example of a concentrated mass acting on the tip of a cantilever beam.

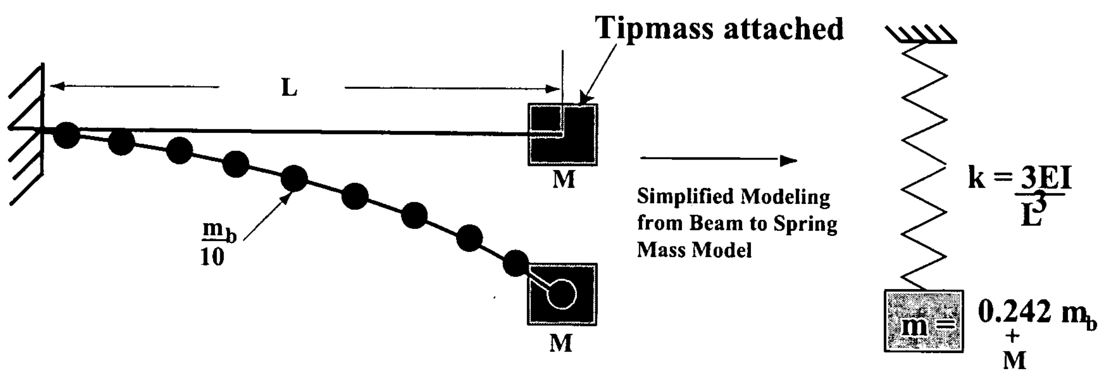


Fig. 5 Modeling Cantilever Beam by an Equivalent Spring-Mass System

The equation of motion for the fundamental mode of the cantilever beam is obtained by adding the concentrated mass to the modal mass derived in (26):

$$m\,\ddot{\delta}(t) + k\,\delta(t) = P, \quad m = 0.242\rho AL + M, \quad k = \frac{3EI_{zz}}{L^3} \tag{28}$$

Note that the spring constant is the same as before.

Fundamental-mode (or first mode) of beam vibrations, with different boundary conditions such as simply supported and built-in beams as well as with other concentrated masses, can be similarly obtained.

B.4 What Are Resonance Frequencies?

Suppose you are driving a car on an isolated test road while varying the speed of your car at will. The test road is constructed to simulate realistically a variety of road conditions, including back

country roads, highways, and freeways. Chances are that you may experience either all or some the following conditions: a sudden rattling of the steering wheel, the seat vibrations, and the vertical motions of the car body itself. If you experience violent vibrations (hopefully luxurious cars don't subject you to such unpleasant vibrations for their money's worth!) from your seat, this vibration can be modeled as follows (see Fig. 1 of Section 2):

$$m\ddot{x}(t) + k\,x(t) = f\ \sin\omega\,t \tag{29}$$

where m is the mass of you and the seat system, k is the spring constant of the seat (and partially the elasticity of your body in seated condition), f is the excitation strength being transmitted from the car floor to the seat frame, and ω is the excitation carrier frequency (or simply excitation frequency).

Dividing (29) by the mass m, we obtain

$$\ddot{x}(t) + p^2\,x(t) \;=\; \frac{f}{m}\,\sin\omega\,t, \qquad p = \sqrt{\frac{k}{m}} \tag{30}$$

where p is called the natural frequency of the spring-mass system, which should not be confused with the excitation frequency ω.

The solution of (30) can be obtained by assuming the following form:

$$x(t) = A\ \sin\omega\,t, \qquad A = \text{constant} \tag{31}$$

Substituting this into (30), we find

$$-\omega^2\,A\ \sin\omega\,t + p^2\,A\ \sin\omega\,t \;=\; \frac{f}{m}\,\sin\omega\,t \quad\Rightarrow\quad (-\omega^2 A + p^2 A - \frac{m}{f})\ \sin\omega t = 0 \tag{32}$$

Since $\sin\omega t$ does not vanish in general, we obtain

$$\begin{aligned} -\,\omega^2\,A\,+p^2\,A \;&=\; \frac{f}{m} \quad\Rightarrow\quad A \;=\; \frac{f/m}{(p^2-\omega^2)} \;=\; \frac{\frac{f}{mp^2}}{1-(\frac{\omega}{p})^2} \;=\; \frac{f}{k}\cdot\frac{1}{1-(\frac{\omega}{p})^2} \\ \text{since}\quad mp^2 &= m\cdot\frac{k}{m} = k \end{aligned} \tag{33}$$

Substituting this into the assumed solution form (31) leads to the following important expression:

$$x(t) = \frac{f}{k}\cdot\frac{1}{1-(\frac{\omega}{p})^2}\ \sin\omega\,t \tag{34}$$

Note that the term $\frac{f}{k}$ in the above equation represents the static deflection x_{st} from

$$k\,x_{st} = f \qquad\Rightarrow\qquad x_{st} = \frac{f}{k} \tag{35}$$

Using this relation, (34) can be written as

$$\frac{x(t)}{x_{st}} = \mathcal{H}(\omega)\sin\omega\, t, \quad \mathcal{H}(\omega) = \frac{1}{1-(\frac{\omega}{p})^2} \tag{36}$$

Observe that, as $|\sin\omega t| \leq 1$, the ratio of the dynamic to the static deflection $\frac{x(t)}{x_{st}}$ is characterized by the *impedance function* $\mathcal{H}(\omega)$.

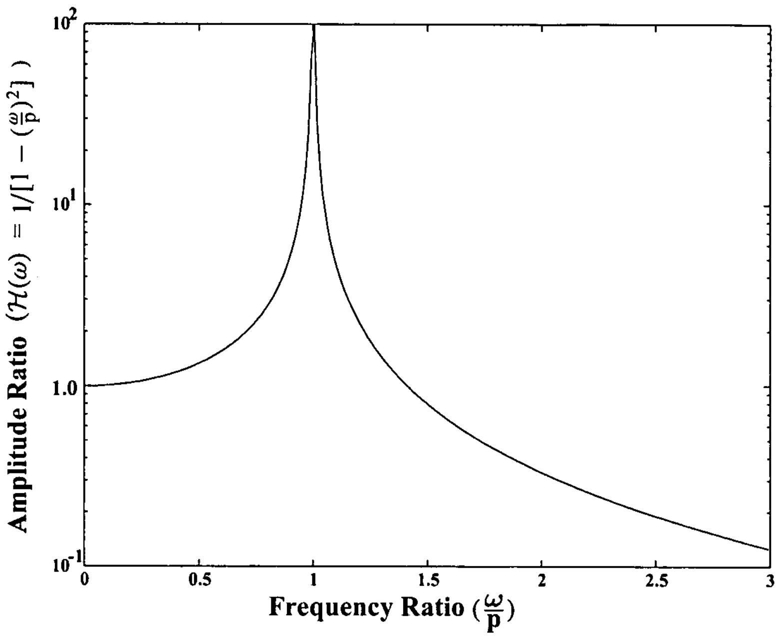


Fig. 6 Effect of Vibration Amplitude as Ratio of Excitation and System Frequencies

Figure 6 illustrates the maximum vibration amplitude as function of the ratio $\frac{\omega}{p}$, that is, the excitation frequency (ω) vs. the natural frequency (p). Note from Fig. 6 that, when $\frac{\omega}{p} = 1$, the absolute magnitude of the system impedance $\mathcal{H}(\omega)$ attains infinity. As the spring cannot be extended to infinity, the mass will have to separate away from the spring. This phenomenon is known as *resonance*.

The foregoing analysis of resonance phenomenon indicates that a good designer must avoid this deleterious happening from the outset. Observe that the system frequency should be either lower, say, $\omega/p \leq 0.5$ or higher $1.25 < \omega/p$ in order to avoid the violent resonant vibrations. In practice, the excitations can vary, suggesting that the natural frequency should be sufficiently away from potential excitation frequency ranges. For example, the excitation frequencies that are transmitted to your car seat from the car floor would not normally consist of a single frequency. Characterization of potential excitation frequencies for specific design components, therefore, constitute a major modeling issue.

It must be reminded that the foregoing single degree of freedom spring-mass system is either an idealization or is concerned with one mode of a structural member. For example, the first three

vibration modes of a cantilever beam is shown in Table 1 taken from Mechanical Vibrations by J. P. Den Hartog.

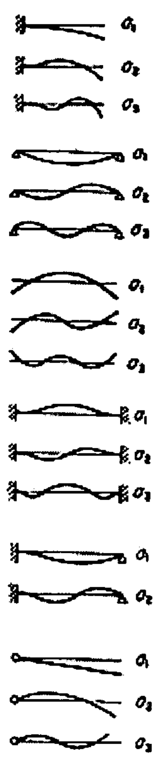
V. Uniform Beams

(Transverse or bending vibrations)

The same general formula holds for all the following cases,

$$\omega_n = a_n \sqrt{\frac{EI}{\mu_1 l^4}}$$

where EI is the bending stiffness of the section, l is the length of the beam, μ_1 is the mass per unit length $= W/gl$, and a_n is a numerical constant, different for each case and listed below



Cantilever or "clamped-free" beam	$a_1 = 3.52$ $a_2 = 22.0$ $a_3 = 61.7$ $a_4 = 121.0$ $a_5 = 200.0$
Simply supported or "hinged-hinged" beam	$a_1 = \pi^2 = 9.87$ $a_2 = 4\pi^2 = 39.5$ $a_3 = 9\pi^2 = 88.9$ $a_4 = 16\pi^2 = 158.$ $a_5 = 25\pi^2 = 247.$
"Free-free" beam or floating ship	$a_1 = 22.0$ $a_2 = 61.7$ $a_3 = 121.0$ $a_4 = 200.0$ $a_5 = 298.2$
"Clamped-clamped" beam has same frequencies as "free-free"	$a_1 = 22.0$ $a_2 = 61.7$ $a_3 = 121.0$ $a_4 = 200.0$ $a_5 = 298.2$
"Clamped-hinged" beam may be considered as half a "clamped-clamped" beam for even a-numbers	$a_1 = 15.4$ $a_2 = 50.0$ $a_3 = 104.$ $a_4 = 178.$ $a_5 = 272.$
"Hinged-free" beam or wing of autogyro may be considered as half a "free-free" beam for even a-numbers	$a_1 = 0$ $a_2 = 15.4$ $a_3 = 50.0$ $a_4 = 104.$ $a_5 = 178.$

Table 1 Modes and Mode Shapes of Beam Vibrations

(Source: Mechanical Vibrations by J. P. Den Hartog, Dover Pub., 1985, p.432)

In order to appreciate further the importance of resonance phenomenon, let us assume that the excitation frequencies ω can be parameterized as

$$\omega = c\sqrt{\frac{EI}{\mu_1\, \ell^4}} \tag{37}$$

where c is a constant parameter.

If $c \approx 22$, then the excitation may not trigger the first mode and third modes. Nevertheless, it would certainly cause the beam to vibrate violently with the second mode. Hence, the designer must consider all the possible vibration modes and modify the design, if necessary, to avoid resonance catastrophes.

B.5 Spring-Mass System with Damper

A well-designed car seat dampens out a wide range of excitations. In addition, there is friction between the seat and the passenger. To model such damping phenomenon, we modify the spring-mass system shown in Fig. 1 by attaching a damper c as shown in Fig. 7.

Single DOF Spring-Mass-Damper Model

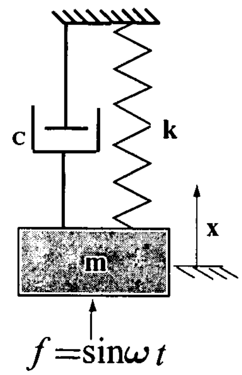


Figure 7 Single DOF Spring-Mass-Damper System

The equation of motion for this system is obtained by modifying that of the undamped system (28) to yield:

$$m\,\ddot{x}(t) + c\,\dot{x}(t) + k\,x(t) = f\sin\omega t \tag{38}$$

It turns out that the solution of the above damped vibration equation can be obtained by

$$x(t) = A\sin\omega t + B\cos\omega t \tag{39}$$

It is recalled from (30) that one needed only the first term of (39) for solving the undamped system (28).

Substituting

$$\begin{aligned} \dot{x}(t) &= \omega(A\cos\omega t - B\sin\omega t) \\ \ddot{x}(t) &= \omega^2(A\sin\omega t + B\cos\omega t) \end{aligned} \tag{40}$$

into (38), one obtains the following equation:

$$(-\omega^2 mA - \omega cB + kA)\ \sin\omega t + (-\omega^2 mB + \omega cA + kB)\ \cos\omega t = f\sin\omega t \tag{41}$$

By equating the lefthand and the right-hand side of the same coefficients of $\sin\omega t$ and $\cos\omega t$, one obtains

$$\begin{aligned} (-\omega^2 mA - \omega cB + kA) &= f \\ (-\omega^2 mB + \omega cA + kB) &= 0 \end{aligned} \tag{42}$$

Equation (42) can be expressed as a 2-by-2 matrix equation:

$$\begin{bmatrix} k-\omega^2 m & -\omega c \\ \omega c & k-\omega^2 m \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} f \\ 0 \end{Bmatrix} \tag{43}$$

In order to simply the resulting solution, we divide the first and second rows of this equation by k to obtain:

$$\begin{bmatrix} (1-\omega^2 \frac{m}{k}) & -\omega \frac{c}{m}\frac{m}{k} \\ \omega \frac{c}{m}\frac{m}{k} & (1-\omega^2 \frac{m}{k}) \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} \frac{f}{k} \\ 0 \end{Bmatrix} \tag{44}$$

Two further substitutions are made to the above matrix. First, we note that

$$\frac{m}{k} = \frac{1}{\frac{k}{m}} = \frac{1}{p^2}, \qquad \Rightarrow \quad \omega^2 \frac{m}{k} = (\frac{\omega}{p})^2 \tag{45}$$

Second, we parameterize c by

$$c = 2\zeta m p, \qquad 0 \leq \zeta \tag{46}$$

so that we have

$$\omega \frac{c}{m}\frac{m}{k} = \frac{c}{m}\omega\frac{m}{k} = 2\zeta p \omega \frac{1}{p^2} = 2\zeta \frac{\omega}{p} \tag{47}$$

Using (45) and (47), (44) is simplified to

$$\begin{bmatrix} [1-(\frac{\omega}{p})^2] & -2\zeta \frac{\omega}{p} \\ 2\zeta \frac{\omega}{p} & [1-(\frac{\omega}{p})^2] \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} \frac{f}{k} \\ 0 \end{Bmatrix} \tag{48}$$

Now, solving for (A, B) we obtain

$$\begin{aligned} A &= \frac{[1-(\frac{\omega}{p})^2]}{[1-(\frac{\omega}{p})^2]^2 + 4\zeta^2(\frac{\omega}{p})^2} \cdot \frac{f}{k} \\ B &= \frac{-2\zeta \frac{\omega}{p}}{[1-(\frac{\omega}{p})^2]^2 + 4\zeta^2(\frac{\omega}{p})^2} \cdot \frac{f}{k} \end{aligned} \tag{49}$$

Finally, if we substitute (A, B) into (38), the resulting solution becomes algebraically quite complex. Therefore, we simplify (38) by using a trigonometric identity as follows. Note that (37) can be modified as

$$\begin{aligned} x(t) = A\sin\omega t + B\cos\omega t &= \sqrt{A^2+B^2}\cdot(\frac{A}{\sqrt{A^2+B^2}}\sin\omega t + \frac{B}{\sqrt{A^2+B^2}}\cos\omega t) \\ &= \sqrt{A^2+B^2}\cdot(\sin\omega t\cos\phi + \sin\phi\cos\omega t) \\ &= \sqrt{A^2+B^2}\cdot sin(\omega t+\phi), \qquad \tan\phi = \frac{B}{A} \end{aligned} \tag{50}$$

Substituting (49) into the above expression, we obtain the solution of the spring-mass-damper system (37) under a sinusoidal excitation as

$$x(t) = \frac{\frac{f}{k}}{\sqrt{[1-(\frac{\omega}{p})^2]^2 + 4\zeta^2(\frac{\omega}{p})^2}} \cdot \sin(\omega t + \phi), \quad \tan\phi = \frac{-2\zeta\frac{\omega}{p}}{[1-(\frac{\omega}{p})^2]}$$

$$\Downarrow \tag{51}$$

$$\frac{x(t)}{x_{st}} = \frac{x(t)}{\frac{f}{k}} = \mathcal{H}\,\sin(\omega t + \phi), \qquad \mathcal{H} = \frac{1}{\sqrt{[1-(\frac{\omega}{p})^2]^2 + 4\zeta^2(\frac{\omega}{p})^2}}$$

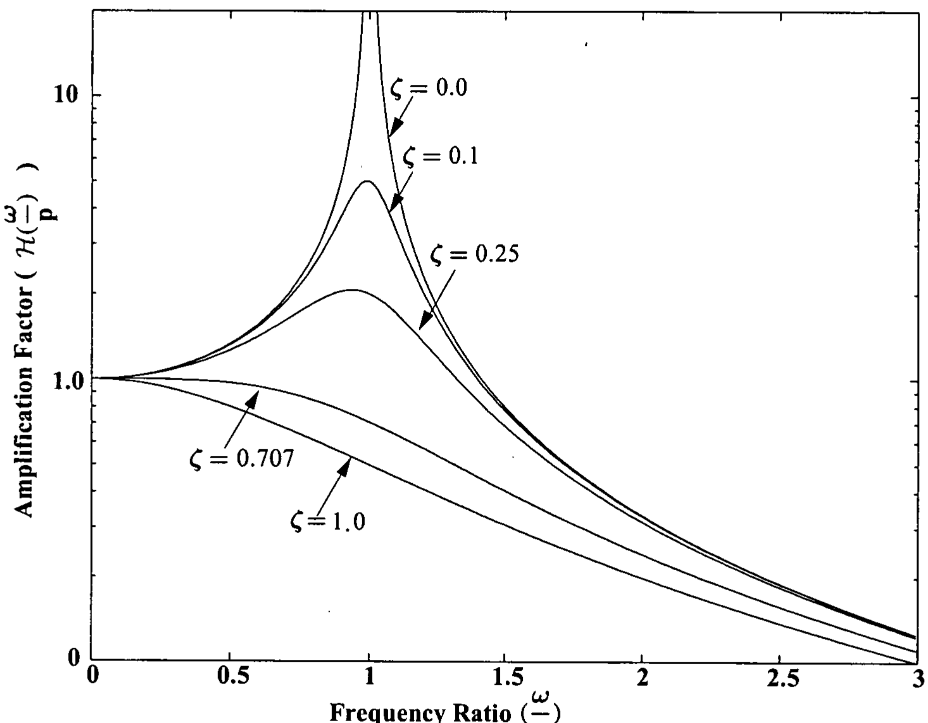


Figure 8 Amplification Factor of Single DOF Spring-Mass-Damper System for Harmonic Excitations

The parameter ζ is called a critical damping ratio and affects the vibration responses as follows:

$$\begin{array}{rl} \text{if} \quad \zeta = 0.0, & \text{the system is undamped (see Eq. (35)) and continues to oscillate;} \\ \text{if} \quad 0 \le \zeta \le 1.0, & \text{the system is damped but eventually reaches to its static displacement;} \\ \text{if} \quad \zeta \ge 1.0, & \text{the system is critically damped and does not oscillate.} \end{array} \tag{52}$$

For a given damping ratio (ζ), the amplification factor $\mathcal{H}$ is uniquely determined by the frequency ratio ($\omega/\mathbf{p}$). This is illustrated in Fig. 8 for various damping ratios. For the undamped case ($\zeta = 0.0$), the result is the same as the one already shown in Fig. 6.

Figure 8 indicates that an ideal choice of the seat design parameters (the seat-and-passenger mass m and the spring constant k_s) is to choose $p = \sqrt{\frac{k_s}{m}}$ to be several times smaller than the largest potential excitation frequency ω. However, this usually implies that the seat spring constant k_s to be impractically small; a heavier passenger may cause the seat frame to buckle! This is where the seat damping can play a major role in mitigating vibration as shown in Fig. 8. Typically, if the seat damping characteristic is chosen such that the resulting damping ratio becomes

$$0.707 \leq \zeta \leq 1.0 \tag{53}$$

it is considered a good compromise (why?). This and additional features to be discussed later in *Class Notes* may serve as a basic design guide for vibration design.

B.6 Predicting the Vibrations of a Real Structure from Laboratory Experiments

Suppose you are tasked to *design* a scaled-down model structure, carry out a laboratory test of its vibrations, and predict the vibrations of a real structure. This is often necessary both for cost, rapid design cycle and facility considerations. For example, it may be feasible to design several full-scale car frames and test them for their vibration characteristics. However, it would not be feasible to design and manufacture several full-scale wings or fuselages, and test them for their vibration characterizations. Table 2 lists a typical scaling relations for correlating a laboratory-model car crash test to a full-scale car crash.

Table 2: Scaling of Laboratory vs. Full-Scale Design Variables
($S \geq 1$: Scale Factor)

Variables	Full Scale	Test System
Length	L	$\ell = L/S$
Displacement	X	x = X/S
Time	T	t = T/S
Velocity	V	V
Stress Intensity Factor	K_c	$k_c = K_c/\sqrt{S}$
Stress	σ	σ
Strain	ϵ	ϵ
Density	ρ	ρ
Young's Modulus	E	E

For the purpose of correlating typical laboratory vibration test data to a full-scale structure, unfortunately, there has not been a straightforward relation for us to employ. One reason for this difficulty is that the frequency equations for plates, shells and cylinder-like structures take on different forms than for the case of beams. By restricting ourselves to the vibrations of beam or beam-like structures, we may utilize the following relation:

$$\frac{\omega_f^2}{\omega_l^2} = \frac{(\frac{EI}{\mu L^4})_f}{(\frac{EI}{\mu L^4})_l} \tag{54}$$

where the subscripts f and l refer to the full-scale and laboratory models, respectively. Nevertheless, many airplane vibration modes can be characterized in terms of simplified beam models.

References

1. R. E. D. Bishop, *Vibration*, Cambridge University Press, 1979.
2. J. P. Den Hartog, *Mechanical Vibrations*, Dover Pub., 1984.